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Scaling in the Ideal Bose Gas

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We have made a detailed study of scaling in the ideal Bose gas in order to resolve the apparent inconsistencies that occur in the scaling laws when the dimensionality of the system is greater than four. We have found that there are not one, but two critical exponents associated with the specific heat singularity that appear in the scaling laws. We have proposed a modification of the scaling laws which is correct in any dimension.

KEY WORDS: Scaling relations; ideal Bose gas; Bose condensation.

1. INTRODUCTION

The scaling hypothesis,(1–5),[2] which has been highly successful in providing a unified approach to the study of equilibrium critical phenomena, predicts certain relations among the critical point exponents. These relations, known as the scaling laws, can be divided into two categories: those relations that are dependent on the dimensionality d, and those relations that are independent of dimensionality. The validity of the dimensionality-dependent laws has sometimes been questioned because of apparent inconsistencies which occur in the ideal Bose gas (or, almost equivalently, in the spherical model), when the dimensionality is greater than four, and because numerical studies show that the critical exponents in the three-dimensional Ising model may

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[2] For a review of scaling see Refs. 6 and 7.

fail to satisfy these dimensionality-dependent laws. In this paper we will only be concerned with the breakdown of scaling when $d > 4$.

The inconsistencies at $d > 4$ have been considered by other authors. Stell$^{(8,9)}$ has proposed a theory of weak scaling in which he argues that the breakdown of scaling in higher dimensions can be interpreted as the emergence of a second correlation length. By introducing a parameter into the scaling relations which describes the behavior of the second correlation length, he is able to make the scaling laws consistent in all dimensions. Domb$^{(10)}$ has suggested that the scaling law $d\nu = 2\Delta + \gamma$ (the critical point exponents ν, Δ, and γ will be defined below), which relates the behavior of correlations to the behavior of thermodynamic functions in the critical region, is an approximation rather than an exact relation and that this approximation becomes worse as the dimensionality increases.

In order to resolve the inconsistencies in the scaling laws when the dimensionality is greater than four, we have made a detailed study of scaling in the ideal Bose gas. We have chosen to study this particular model, which is virtually equivalent to the spherical model in its properties near the critical point, because it is one of the few models possessing a first-order phase transition which is exactly solvable in all dimensions. It is also a model for which the scaling relations hold exactly when $d < 4$ but for which the scaling relations break down when $d > 4$.

In this paper our approach will be to consider the physical arguments that have been put forth as a basis for scaling and which predict the scaling relations.[3] We will then examine the validity of these arguments, compare them in detail with the known properties of the Bose gas, and see why they are inapplicable when $d > 4$. We will then propose a modification of the scaling laws which is in accordance with what we have learned from the ideal Bose gas, and is consistent in all dimensions.

The critical exponents that we will use are defined in Table I and the scaling relations that we will discuss are listed in Table II. Since these relations are not all independent, one can examine an independent subset from which the others can be derived. We have chosen to study the subset consisting of the first four relations primarily because each one of them may be obtained from basic theoretical arguments. The fifth relation, which is a dependent relation, is also derivable on the basis of a theoretical argument and will also be examined in detail.

In Section 2 we will describe the physical arguments that underlie the first five scaling relations on Table I and in Section 3 we will review those properties of the ideal Bose gas that are necessary for our discussion. Readers familiar with the scaling arguments and with the ideal Bose gas may proceed to Section 4, where we apply these arguments to the ideal Bose gas.

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[3] We will not discuss scaling from the renormalization-group point of view.

Table I. Definition of Critical Exponents[a]

Exponent	Definition	
α	$C_h \sim t^{-\alpha}$	$t \to 0+,\ h = 0$
β	$m \sim (-t)^{\beta}$	$t \to 0-$, coexistence curve
δ	$h \sim \lvert m\rvert^{\delta}\,\mathrm{sgn}(m)$	$t = 0$
γ	$\chi_T \sim t^{-\gamma}$	$t \to 0+,\ h = 0$
ν	$\xi \sim t^{-\nu}$	$t \to 0+,\ h = 0$
η	$G(r) \sim r^{-(d-2+\eta)}$	$r \to \infty,\ t = 0$
Δ	$\partial\mathcal{F}/\partial h \sim (-t)^{-\Delta}$	$t \to 0-,\ h = 0$

[a] Here m is the magnetization, h is the magnetic field, t is the reduced temperature, C_h is the specific heat at constant field, χ_T is the isothermal susceptibility, ξ is the correlation length, $G(r)$ is the pair correlation function, and $\mathcal{F}$ is the Gibbs free energy.

2. THE THEORETICAL ARGUMENTS WHICH PREDICT THE SCALING RELATIONS

We will now review the general theoretical arguments that have been used in the past to justify the first five scaling relations on Table I. In describing these arguments we will switch back and forth between fluid language and magnetic language (see Ref. 7), depending on what is the most convenient way to state these arguments and what relates most directly to the Bose gas.

2.1. The Relations $\delta = 1 + \gamma/\beta$ and $\alpha + 2\beta + \gamma = 2$; Homogeneity or Scaling Hypothesis for the Free Energy

The scaling relations $\delta = 1 + \gamma/\beta$ and $\alpha + 2\beta + \gamma = 2$ may be obtained from a homogeneity or scaling hypothesis for the thermodynamic free energy.$^{(1-7)}$ The hypothesis is that the singular part of the free energy becomes a generalized homogeneous function of the magnetic field h and the reduced temperature t in the vicinity of the critical point. Recall that a generalized homogeneous function is defined$^{(7)}$ as one which has the following property:

$$f(\lambda^a x, \lambda y) = \lambda^p f(x, y)$$

Table II. The Scaling Relations

1. $\delta = 1 + \gamma/\beta$
2. $\alpha + 2\beta + \gamma = 2$
3. $(2 - \eta)\nu = \gamma$
4. $d\nu = 2 - \alpha$
5. $d\nu - \gamma = 2\beta$
6. $\nu(d - 2 + \eta) = 2\beta$
7. $2 - \eta = d(\delta - 1)/(\delta + 1) = d\gamma/(2\beta + \gamma)$
8. $d - 2 + \eta = 2d/(\delta + 1) = 2d\beta/(2 - \alpha)$

It can be shown that if f is a generalized homogeneous function of its variables, then it may be written in the form

$$f(x, y) = y^p F\left(\frac{x}{y^a}\right) = x^{p/a}\tilde{F}\left(\frac{y}{x^{1/a}}\right)$$

The homogeneity hypothesis says that if we take the free energy and subtract off all nonsingular terms, then the remaining part $\mathscr{F}_s(h, t)$ will have the following form near the critical point:

$$\mathscr{F}_s(h, t) = t^p F\left(\frac{h}{t^a}\right) = h^{p/a}\tilde{F}\left(\frac{t}{h^{1/a}}\right) \tag{1}$$

where a and p are numbers and F is an undetermined function. From this we can obtain homogeneous forms for the magnetization, susceptibility, and specific heat by differentiating with respect to the appropriate variable:

$$\begin{aligned} M(h, t) &= \partial\mathscr{F}_s(h, t)/\partial h \\ \chi(h, t) &= \partial^2\mathscr{F}_s(h, t)/\partial h^2 \\ C(h, t) &= \partial^2\mathscr{F}_s(h, t)/\partial T^2 \end{aligned}$$

We can then determine the exponents α, β, γ, and δ in terms of p and a by comparing the resulting forms with the exponent definitions (see Table I). If we eliminate p and a from our expressions for the exponents, we obtain the scaling relations

$$\delta = 1 + (\gamma/\beta), \qquad \alpha + 2\beta + \gamma = 2$$

It is important to keep in mind that the specific heat exponent which enters this second relation is defined as

$$\frac{\partial^2}{\partial t^2}\left[\mathscr{F}(h, t) - \mathscr{F}_{\text{nonsingular}}(h, t)\right] \sim t^{-\alpha_h} \tag{2}$$

where $\mathscr{F}$ is the free energy and $\mathscr{F}_{\text{nonsingular}}$ is the nonsingular part of the free energy. Here we have put a subscript h on the exponent α in order to remind ourselves that this exponent is associated with a homogeneity argument. When the specific heat exponent is defined in this way, the second scaling relation will be satisfied if $\mathscr{F}_s(h, t)$ is a homogeneous function of its variables. To emphasize this, we may rewrite the second scaling relation as[4]

$$\alpha_h + 2\beta + \gamma = 2$$

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[4] Strictly speaking, these two scaling relations should be $\alpha_h' + 2\beta + \gamma' = 2$ and $\delta = 1 + \gamma'/\beta$, where a prime indicates that the exponent is defined for $T < T_c$. However, a consequence of the homogeneity hypothesis is that the exponents above and below T_c must be equal, so that the relations as they are written without the primes are correct. One exception that might occur is that the second derivative with respect to h

It is also worthwhile noting that there are a number of equivalent ways of formulating a homogeneity hypothesis for the thermodynamic variables and thereby obtaining these relations.(1–5) In particular, it can be shown that if the magnetic field is a homogeneous function of the magnetization and reduced temperature, then the first two relations will be satisfied.

2.2. The Relation $\gamma = (2 - \eta)\nu$; Homogeneity Hypothesis for the Correlation Function

The relation $\gamma = (2 - \eta)\nu$ can also be derived on the basis of a scaling or homogeneity hypothesis.(11) This homogeneity hypothesis concerns the pair correlation function $G(r, t)$ in zero field and says that $G(r, t)$ is a generalized homogeneous function at large distances and in the vicinity of the critical point, and therefore that it may be written in the scaling form

$$G(r, t) \sim r^{-(d-2+\eta)} g(r/\xi), \qquad r \gg r_0 \tag{3}$$

where ξ is the correlation length, $\xi \sim t^{-\nu}$, r_0 measures the range of the dominant interactions, and the function $g(r/\xi)$ satisfies certain conditions.[5] To obtain the relation $\gamma = (2 - \eta)\nu$, we note that the susceptibility is given by

$$\chi = \int G(r, t)\, d\mathbf{r}$$

where $d\mathbf{r}$ is a volume element and the integral runs over all space. Since the susceptibility diverges as the critical point is approached and the integrand $G(r, t)$ is bounded, we can conclude that the dominant contribution to the integral comes from the behavior of $G(r, t)$ at large distances. But then $G(r, t)$ in the integral may be replaced by its asymptotic form [Eq. (3)] and the result is an asymptotic formula for χ that holds in the neighborhood of the critical point. It is found that

$$\chi \sim \xi^{2-\eta}$$

which implies that

$$\gamma = (2 - \eta)\nu$$

This is often referred to as Fisher's relation.

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of the function $F(h/t^{\beta\delta})$ in Eq. (1) is identically equal to zero for $T < T_c$ and is variable for $T > T_c$. In this case the exponent γ' would not be defined. This is, in fact, what happens in the ideal Bose gas when $d < 4$. One still expects, however, that the relations $\alpha_h + 2\beta + \gamma = 2$ and $\delta = 1 + \gamma/\beta$ will be satisfied, although this will not be guaranteed.

[5] The conditions are that $g(0)$ be finite and nonzero and that $g(x) \sim x^{\frac{1}{2}(d-3)+\eta} e^{-x}$ as $x \to \infty$.

2.3. The Relation $d\nu = 2 - \alpha$

The theoretical arguments(12) that are usually cited as justification for this relation are the following: The approach to the critical point is characterized by a divergence in ξ, the correlation range of spontaneous density fluctuations, and by a divergence in C_v, the specific heat at constant volume. It is reasonable to assume that these two phenomena are closely related and share a common origin. This assumption leads to the conclusion that the free energy density associated with a divergence in ξ should have the same magnitude as the free energy density associated with the divergence in C_v. It is further assumed that the free energy associated with the divergence in C_v is proportional to $t^{2-\alpha}$ and that the free energy density associated with the divergence in ξ is proportional to $kT|\xi^d$. Since ξ diverges as $t^{-\nu}$ near the critical point we obtain the relation $d\nu = 2 - \alpha$. This is called the hyperscaling relation.

2.4. The Relation $d\nu = \gamma + 2\beta$

This relation can be derived by combining the first four relations in Table II. It is of interest, however, because it can also be derived independently on the basis of general theoretical arguments(12) which we now describe. We consider a subvolume v in the interior of a bulk homogeneous fluid which is in equilibrium with its conjugate phase near the critical point. It is postulated that density fluctuations of the order of the density difference between the two equilibrium phases occur with high probability when the size of the subvolume becomes equal to ξ^d. This implies that the mean-square fluctuation in the volume ξ^d is given by

$$(\Delta\rho_{\xi^d})^2 \sim |\rho_l - \rho_g|^2 \sim t^{2\beta} \tag{4}$$

where ρ_l and ρ_g are the densities of the two phases in equilibrium and β is the exponent describing the shape of the coexistence curve as defined in Table I. An alternative calculation of the mean-square density fluctuation yields the fifth relation as follows: The mean-square density fluctuation can be obtained in terms of the pair correlation function $G(r, t)$ by assuming that the equation

$$(\Delta\rho_v)^2 = (1/v)\int_v G(r, t)\, d\mathbf{r} = \rho^2 kT\chi/v$$

is correct even when v is the microscopic volume ξ^d. The resulting expression $(\Delta\rho_{\xi^d})^2$ will be asymptotically correct in the neighborhood of the critical point and gives

$$(\Delta\rho_{\xi^d})^2 \sim t^{d\nu-\gamma}$$

Comparing this with Eq. (4) gives

$$2\beta = d\nu - \gamma$$

which is the fifth relation.

3. THE THERMODYNAMIC PROPERTIES OF THE IDEAL BOSE GAS

The thermodynamic properties of the ideal Bose gas in the neighborhood of its transition have been studied in detail by Gunton and Buckingham,(14,15) who viewed the condensation as a cooperative transition. They introduced an order parameter called the Bose moment ψ_0, which is equal to the square root of the density of particles in the gound state (the condensate), and an ordering field μ_0, called the Bose field, which is the intensive variable conjugate to the Bose moment. The Bose moment and Bose field are analogous to the magnetic moment and magnetic field of a system which undergoes a ferromagnetic phase transition. As in the ferromagnetic case, the transition takes place only in zero field.

We will first review Gunton and Buckingham's derivation of the critical exponents from the asymptotic behavior of the equation of state in the neighborhood of the transition. These critical exponents for the Bose gas along with their definitions are listed in Table III. We will see that this set of exponents, which comes from an exactly soluble model, does *not* satisfy the scaling relations when $d > 4$. This will lead us in Section 4 to examine more closely the physical arguments which led to these scaling relations.

Gunton and Buckingham found that the pressure of the ideal Bose gas is given by

$$P(\mu, \mu_0, T) = -\mu_0{}^2/\mu + (\beta\lambda^d)^{-1}F_{d/2+1}(-\beta\mu) \tag{5}$$

where μ is the chemical potential and $\lambda^2 = (2\pi h/mkT)^d$. The function F is

Table III. Critical Exponents of the Ideal Bose Gas

Exponent	Definition	Conditions	$d < 4$	$d > 4$
$\alpha_s{}^a$	$C_{\rho,\mu_0} \sim t^{-\alpha_s}$	$t \to 0+, \mu_0 = 0$	$-(4-d)/(d-2)$	$-(d-4)/2$
β	$\psi_0 \sim (-t)^\beta$	$t \to 0+,$ coexistence curve	$\frac{1}{2}$	$\frac{1}{2}$
δ	$\mu_0 \sim \psi_0{}^\delta$	$t = 0$	$(d+2)/(d-2)$	3
γ	$\chi = kT\, \partial\psi_0/\partial\mu_0$	$t \to 0+, \mu_0 = 0$	$2/(d-2)$	1
ν	$\xi \sim t^{-\nu}$	$t \to 0+, \mu_0 = 0$	$1/(d-2)$	$\frac{1}{2}$
η	$G_B(r) \sim r^{-(d-2+\eta)}$	$r \to \infty, t = 0, \mu_0 = 0$	0	0

[a] The critical exponent α_s is more explicitly defined in Eq. (12).

defined by

$$F_\sigma(x) = \sum_{n=1}^{\infty} n^{-\sigma} e^{-nx}, \qquad x \geqslant 0$$

and has the following asymptotic expansion when its argument is small[16]:

$$F_\sigma(x) = \Gamma(1 - \sigma)x^{\sigma-1} + \sum_{n=1}^{\infty} (-1)^n \zeta(\sigma - n)x^n/n! \tag{6}$$

where ζ is the Riemann zeta function. They also found that the Bose moment is given by

$$\psi_0 = -\mu_0/\mu \tag{7}$$

and the density is given by

$$\rho = (\mu_0/\mu)^2 + (T/T_c)^{d/2} F_{d/2}(-\beta u)/F_{d/2}(0) \tag{8}$$

where the critical temperature T_c is a function of

$$\rho,\ T_c(\rho) = 2\pi h^2/mk[\rho^{-1}F_{d/2}(0)]^{2/d}$$

Equation (8) is a conservation equation for the density, with $\mu_0{}^2/\mu^2 = \psi_0{}^2$ representing the density of particles in the ground state. The equation of state is found by solving this equation for $\mu_0(\psi_0, T)$ and it can be shown that a transition occurs only when $\mu_0 = 0$, as shown in Fig. 1. If we compare Fig. 1 with the phase diagram of a ferromagnet in the h–t plane, we can see that an obvious analogy exists between the two systems.

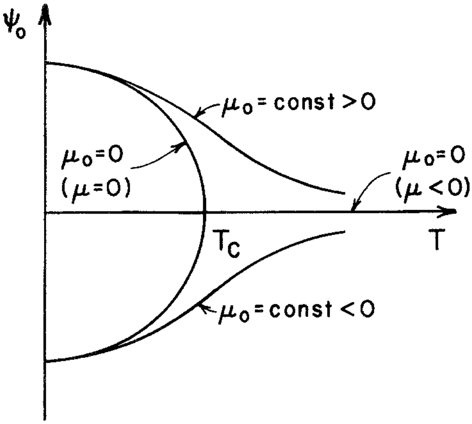

Fig. 1. The equation of state of the ideal Bose gas is shown schematically in the temperature–density plane. The overall density ρ is fixed at a constant. This is analogous to the phase diagram of a ferromagnet, with the Bose moment and Bose field being analogous, respectively, to the magnetic moment and magnetic field.

Near the transition point where μ_0, ψ_0, μ, and $t = [T - T_c(\rho)]/T_c(\rho)$ are small, the known expansions of the function F [Eq. (6)] are used to obtain the asymptotic behavior of the equation of state. The equation of state will depend on whether d is less than, equal to, or greater than four as follows:

$$\mu_0/kT_c\psi_0 \sim \begin{cases} (\frac{1}{2}dt + \rho^{-1}\psi^2)^{2/(d-2)}, & d < 4 \\ (\frac{1}{2}dt + \rho^{-1}\psi^2)/\ln(\frac{1}{2}dt + \rho^{-1}\psi^2)^{-1}, & d = 4 \\ (\frac{1}{2}dt + \rho^{-1}\psi^2), & d > 4 \end{cases} \tag{9}$$

Gunton and Buckingham also calculated the Bose moment correlation function $G_B(r, t)$ and found that it has the following form for large r and in the vicinity of the critical point (large ξ):

$$G_B(r, t) \sim r^{-(d-2)}(r/\xi)^{(d/2)-1}K_{(d/2)-1}(r/\xi) \tag{10}$$

where K is a Bessel function of the second kind and ξ is the correlation length and is identified as

$$\xi = (-\beta\mu)^{-1/2} = (\beta\mu_0/\psi_0)^{-1/2} \tag{11}$$

The critical exponents found by Gunton and Buckingham are summarized in Table III. They are all defined for fixed density, and with the exception of β, for $T > T_c$. When the dimensionality is less than four the exponents are dimensionality dependent but when the dimensionality is greater than four the exponents (with the exception of α_s) become dimensionality independent and take on their mean-field values. When $d = 4$ the exponents are the same as for $d > 4$ but there are also logarithmic factors to be considered. In this paper we will not discuss the borderline dimension $d = 4$.

The specific heat C_{ρ,μ_0} remains finite as $t \to +0$ but one of its temperature derivatives may diverge as $t \to +0$. Accordingly, the exponent α_s is defined such that the first temperature derivative of C_{ρ,μ_0} to diverge is the mth one, with

$$\partial^m C/\partial T^m \sim t^{-\alpha_s - m} \tag{12}$$

Thus α_s characterizes the dominant singularity in the specific heat, and corresponds to the usual definition of the specific heat exponent when α_s is positive.

4. THE APPLICATION OF THE SCALING ARGUMENTS TO THE IDEAL BOSE GAS

We may now ask to what extent the scaling relations are satisfied by the set of exponents listed in Table III for the ideal Bose gas. If we substitute these values for the exponents into the first four relations on Table II, we see that they are all satisfied when the dimensionality is less than four but that when the dimensionality is greater than four the second relation no longer holds, i.e.,

$$\alpha_s + 2\beta + \gamma \neq 2 \qquad \text{when} \quad d > 4$$

for the ideal Bose gas. Naturally the remaining relations in Table II, which can be obtained by combining relations 1–3 with 4, will not be satisfied when $d > 4$.

An examination of the equation of state for the Bose gas, Eq. (9), reveals that is has a homogeneous form and that therefore the relations $\delta = 1 + (\gamma/\beta)$ and $\alpha_h + 2\beta + \gamma = 2$, where α_h is defined in Eq. (2), must be satisfied. The relation $\delta = 1 + (\gamma/\beta)$ is satisfied by the Bose gas in all dimensions. Since $\alpha_s + 2\beta + \gamma \neq 2$ when $d > 4$ in the Bose gas we must conclude that

$$\alpha_h \neq \alpha_s \qquad \text{when} \quad d > 4$$

and that

$$\alpha_h = \alpha_s \qquad \text{when} \quad d < 4$$

In order to see why α_s and α_h are different when $d > 4$, we must look at the free energy of the ideal Bose gas in the neighborhood of the critical point. We consider the free energy

$$\mathscr{F}(\rho, \mu_0, T) = P(\mu, \mu_0, T) - \rho\mu$$

and hold the density fixed so that it becomes a function only of the Bose moment and temperature. This free energy is analogous to the free energy per spin $\mathscr{F}(h, t)$ of a magnetic system in the sense that it has analogous variables but with renormalization of exponents because of fixed p.(17) It may be expanded near the critical point in terms of the inverse correlation length by using Eqs. (5), (6), and (11),

$$\begin{aligned}\mathscr{F}(\rho, \mu_0, T) = \mathscr{F}(\mu_0, T) \sim \mu_0\psi_0 &+ (\beta\lambda^d)^{-1}[\xi^{-d}\Gamma(d/2) + \zeta(d/2) \\ &+ \sum_{n=2} (-1)^n \xi^{2n}\zeta(\tfrac{1}{2}d - 1 - n)/n!] \\ &+ (\beta\lambda^d)^{-1}\zeta(d/2)[(\lambda_c/\lambda)^d - 1]\xi^{-2}\end{aligned}$$

or, more simply,

$$\begin{aligned}\mathscr{F}(\mu_0, T) \sim \mu_0\psi_0 &+ (\beta\lambda^d)^{-1}[\text{const} + O(\xi^{-d}) + O(\xi^{-4}) + O(\xi^{-2}t) \\ &+ O(\xi^{-6}) + \cdots]\end{aligned}$$

It can easily be shown that the term $\sim\xi^{-2}t$ is no larger than the term $\sim\xi^{-4}$.

In order to determine the exponent α_h for the Bose gas, we must calculate the second derivative of the quantity $\mathscr{F}(\mu_0, T) - \mathscr{F}_{\text{nonsingular}}(\mu_0, T)$ and evaluate it along the critical isochore (since we are calculating C_{ρ,μ_0} at $\mu_0 = 0$). Near the critical point

$$\begin{aligned} \mathscr{F}(\mu_0, T) - \mathscr{F}_{\text{nonsingular}} &\sim (\beta\lambda^d)^{-1}[\xi^{-d} + \xi^{-4} + \cdots] \\ &\sim [t^{d\nu} f_1(\mu_0/t^{\beta\delta}) + t^{4\nu} f_2(\mu_0/t^{\beta\delta})] \end{aligned} \tag{13}$$

to leading order, where f_1 and f_2 are functions which are finite at ∞ and where we have dropped the multiplicative constants. We have also dropped the term $\sim\mu_0\psi_0$ since it will not contribute anything to the specific heat when $\mu_0 = 0$.

We can see immediately that a competititon exists between these two terms in the free energy so that when $d < 4$ the first term will dominate and when $d > 4$ the second term will dominate. The specific heat will be

$$\begin{aligned} C - C_{\text{nonsingular}} &= \partial^2[\mathscr{F}(\mu_0, T) - \mathscr{F}_{\text{nonsingular}}]/\partial T^2 \\ &\sim t^{d\nu-2} f_1(\mu_0/t^{\beta\delta}) + t^{4\nu-2} f_2(\mu_0/t^{\beta\delta}) \end{aligned} \tag{14}$$

so that on the critical isochore

$$\begin{aligned} C - C_{\text{nonsingular}} &\sim t^{d\nu-2} f_1(0), \qquad && d < 4 \\ &\sim t^{4\nu-2} f_2(0), && d > 4 \end{aligned}$$

From this it follows that the exponent α_h, which is determined by the dominant term in Eq. (14), will be

$$\begin{aligned} \alpha_h &= d\nu - 2 = -(4 - d)/(d - 2), \qquad && d < 4 \\ &= 4\nu - 2 = 0, && d > 4 \end{aligned}$$

where we have inserted the values $\nu = 1/(d - 2)$ for $d < 4$ and $\nu = \frac{1}{2}$ for $d > 4$ from Table III.

We can also obtain the exponent α_s [defined in Eq. (12)] from Eq. (14) and we find that

$$\begin{aligned} \alpha_s = d\nu - 2 &= -(4 - d)/(d - 2), \qquad && d < 4 \\ &= (4 - d)/2, && d > 4 \end{aligned}$$

That is, if we differentiate Eq. (14) until one of the terms diverges, we will find that it is always the first term which contains the divergence.[6]

We see then that when $d < 4$, the dominant term in $\mathscr{F}_s(\mu_0, T)$ (which is what the exponent α_h picks out) will be the same as the leading term singular in the temperature (which is what the exponent α_s picks out). However, when $d > 4$ the dominant term in Eq. (13) will *not* be the same as the leading term singular in the temperature. The exponents α_h and α_s therefore describe

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[6] This occurs even when d is an even integer. This is because the first two terms in the expansion for F [see Eq. (6)] diverge separately when σ is an integer but combine to yield a term proportional to $x^{\sigma-1} \log x$. See Ref. 16. In this case the term proportional to $t^{d\nu-2}$ in the free energy would be multiplied by a term of the order of $\log t$.

different aspects of the behavior of the specific heat and they cannot in general be used interchangeably.

One question that might be raised is whether the second term in Eq. (14), which is proportional to $t^0 f_2(\mu_0/t^{\beta\delta})$ when $d > 4$, should be included in the term $C_{\text{nonsingular}}$ on the left-hand side of Eq. (14), since at first sight it appears to be nonsingular. It can be shown, however, that although this term is nonsingular in the temperature variable, the function f_2 contains a singularity with respect to the field variable μ_0 and is, in fact, the origin of the susceptibility divergence when $d > 4$. Therefore it should not be included in the term $C_{\text{nonsingular}}$ and our determination of α_h is correct.

We can ask further whether it is the exponent α_h or α_s which belongs in the relation $d\nu = 2 - \alpha$. By inspection of Eqs. (13) and (14) we can see that the term in the free energy that is proportional to ξ^{-d} is always the term that contains the leading singularity with respect to the temperature variable, so that this term will also be proportional to $t^{2-\alpha_s}$. Thus the fourth relation on Table II should read

$$d\nu = 2 - \alpha_s$$

The reasoning that we quoted earlier for the fourth relation $d\nu = 2 - \alpha$ is now seen to be correct only if one recognizes that it is the term in the free energy density associated with the exponent α_s that is proportional to kT/ξ^d for all d rather than the free energy density itself which is proportional to kT/ξ^d.

The relation $\gamma = (2 - \eta)\nu$ is always satisfied by the ideal Bose gas. This is not surprising since this relation was obtained by assuming that the correlation function has a homogeneous form and the Bose gas does indeed have the postulated homogeneous form [compare Eqs. (3) and (10)].

We have seen that the first four scaling relations in Table II should read

$$\delta = 1 + \gamma/\beta, \qquad \alpha_h + 2\beta + \gamma = 2, \qquad (2 - \eta)\nu = \gamma, \qquad d\nu = 2 - \alpha_s$$

When the dimensionality is less than four $\alpha_h = \alpha_s$, so that we can just call these two exponents α, as is normally done, and we can then combine these relations to give us the rest of the relations in Table I. However, when $d > 4$, $\alpha_h \neq \alpha_s$ and therefore any relations which were derived by combining the first four relations and assuming that α_h and α_s are indistinguishable will not be valid for all dimensions. The correct form for the last four scaling relations in Table II include a term proportional to $\alpha_h - \alpha_s$ as follows:

$$\begin{aligned}
d\nu - \gamma &= 2\beta + (\alpha_h - \alpha_s) \\
\nu(d - 2 + \eta) &= 2\beta + (\alpha_h - \alpha_s) \\
2 - \eta &= d(\delta - 1)/[\delta + 1 + (\alpha_h - \alpha_s)/\beta] \\
&= d\gamma/(2\beta + \gamma + \alpha_h - \alpha_s) \\
d - 2 + \eta &= [2 + (\alpha_h - \alpha_s)\, d/\beta]/[\delta + 1 + (\alpha_h - \alpha_s)/\beta] \\
&= [2d\beta + (\alpha_h - \alpha_s)\, d]/(2 - \alpha_s)
\end{aligned}$$

This leaves us with the problem of explaining why the physical arguments described previously for the fifth relation are in error when $d > 4$ and how they can be amended to include the extra $\alpha_h - \alpha_s$. The solution to this problem is elusive because the specific heat does not enter into the argument at all. For the Bose gas, the point at which the argument breaks down is the assumption $(\Delta\rho_{\xi^d})^2 \sim (\rho_l - \rho_g)^2$ [Eq. (4)] rather than the assumption $(\Delta\rho_{\xi^d})^2 \sim t^{d\nu-\gamma}$. We find that the mean square density fluctuations in a volume of size $\sim \xi^d$ are smaller than $(\rho_l - \rho_g)^2$ or more specifically

$$(\Delta\rho_{\xi^d})^2 \sim (\rho_l - \rho_g)^{2+(\alpha_h-\alpha_s)/\beta}$$

Thus, for the ideal Bose gas we have confirmed Stell's conclusion that [Ref. 9, especially Eqs. (11), (15), and (16)]

$$(\Delta\rho_{\xi^d})^2 \sim (\rho_l - \rho_g)^{2+p}$$

with $p > 0$ in higher dimensions. Use of our $2\beta + (\alpha_h - \alpha_s)/\beta = \nu(d - 2 + \eta)$ further confirms his $p = \varepsilon(d - 2 + \eta)$, where $\varepsilon = \nu/\beta$ in the Bose gas. In this context, Stell defines his "second length" as the Λ for which

$$(\Delta\rho_{\Lambda^d})^2 = (\rho_l - \rho_g)^2$$

is satisfied.

SUMMARY AND CONCLUSIONS

We have sought to understand why certain of the scaling relations are not satisfied by the ideal Bose gas (and therefore by the spherical model) when the dimensionality is greater than four. We have found that this occurs because of a competitition that exists between two leading terms in the free energy

$$\mathscr{F}(\mu_0, T) - \mathscr{F}_{\text{nonsingular}} \sim t^{d\nu} f_1(\mu_0/t^{\beta\delta}) + t^{4\nu} f_2(\pi_0/t^{\beta\delta})$$

as the critical point is approached. When the dimensionality is less than four, the term proportional to $t^{d\nu}$ dominates, as is consistent with the assumption (see Section 2.3) that the free energy be proportional to kT/ξ^d. When the dimensionality is greater than four, the term proportional to $t^{4\nu}$ dominates and the former assumption is no longer correct.[7]

A careful examination of the physical arguments reveals that the scaling relations will continue to hold even when $\mathscr{F}$ is not proportional to kT/ξ^d, if we recognize that there are not one, but two specific heat exponents which belong in the scaling relations. These two exponents, which we have called α_s and α_h, describe different aspects of the specific heat and are associated with different terms in the free energy when $d > 4$. Each exponent has its own role

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[7] A similar argument that involves a related competition in $G(r, h, t)$ for fixed small r is given by Stell.(18)

in the scaling relations; the exponent α_h belongs in the scaling relation $\alpha_h + 2\beta + \gamma = 2$ and the exponent α_s belongs in the scaling relation $d\nu = 2 - \alpha_s$. When the dimensionality is less than four, the exponents α_h and α_s are equal since they are associated with the same term in the free energy and therefore the scaling relations take on the usual form with a single specific heat exponent α (as given in Table I). When the dimensionality is greater than four, the exponents α_h and α_s are not equal since they are associated with different terms in the free energy and therefore the scaling relations take the form given below.

The exponents α_h and α_s are defined in the following way: If the first temperature derivative of the specific heat to diverge is the mth one, then α_s is given by

$$\partial^m C/\partial T_m \sim t^{-\alpha_s - m}$$

The exponent α_h is defined as

$$\partial^2(\mathscr{F}(\mu_0, T) - \mathscr{F}_{\text{nonsingular}})/\partial T^2 \sim t^{-\alpha_h}$$

One can see therefore that the exponent α_s picks out the leading term in the specific heat that is singular with respect to the temperature, while the exponent α_h just picks out the leading term in the singular part of the specific heat. Here, by a singular function we mean that one of the function's derivatives with respect to either the temperature or the field (or both) diverges. When $d > 4$, the leading term in the singular part of the specific heat is not the same as the leading term in the specific heat that is singular with respect to the temperature. This is because the leading term in the singular part of the specific heat is analytic in the temperature but singular in the field. For this reason α_s and α_h are not the same when $d > 4$.

The correct form for the scaling relations is

$$\delta = 1 + \gamma/\beta, \qquad \alpha_h + 2\beta + \gamma = 2, \qquad (2 - \eta)\nu = \gamma, \qquad d\nu = 2 - \alpha_s$$

$$d\nu - \gamma = 2\beta + (\alpha_h - \alpha_s), \qquad \nu(d - 2 + \eta) = 2\beta + (\alpha_h - \alpha_s)$$

$$\begin{aligned} 2 - \eta &= d(\delta - 1)/[\delta + 1 + (\alpha_h - \alpha_s)/\beta] \\ &= d\gamma/[2\beta + \gamma + \alpha_h - \alpha_s] \\ d - 2 + \eta &= [2 + (\alpha_h - \alpha_s)\, d/\beta]/[\delta + 1 + (\alpha_h - \alpha_s)/\beta] \\ &= [2d\beta + d(\alpha_h - \alpha_s)]/(2 - \alpha_s) \end{aligned}$$

This is consistent in all dimensions and is correct for the Bose gas. We think that it (or appropriate generalizations) will also be satisfied by any of the models which have heretofore been thought to violate scaling in greater than four dimensions. This includes the spherical model, the spherical model with long-range interactions, and Ornstein–Zernike systems.(18,19)

Finally, we would like to comment on the relationship between the scaling form of the correlation function and the scaling form of the free energy for the Bose gas. It can be shown in general(21) that if the correlation function in nonzero field $G(r, h, t)$ has a scaling form, then the free energy $\mathcal{F}_s(h, T)$ will also have a scaling form. More explicitly, if $G(r, h, T)$ has the scaling form

$$G(r, h, T) \sim r^{-(d-2+n)} D(r/t^{-\nu}, h/t^{\beta\delta})$$

then the free energy will have the scaling form

$$\mathcal{F}_s(h, T) \sim t^{2-\alpha_h} W(h/t^{\beta\delta})$$

where $2 - \alpha_h = 2\beta + \gamma = \beta(\delta + 1)$. This can be seen by integrating $G(r, h, T)$ once with respect to r to obtain the susceptibility and then by integrating twice with respect to h to obtain the free energy. On the other hand, in the Bose gas when $d > 4$, the free energy $\mathcal{F}_s(\mu_0, T)$ has a scaling form which contains a term proportional to $t^{2-\alpha_s}$ in addition to the expected term proportional to $t^{2-\alpha_h}$, i.e.,

$$\mathcal{F}_s(\mu_0, T) \sim t^{2-\alpha_h} W_1(\mu_0/t^{\beta\delta}) + t^{2-\alpha_s} W_2(\mu_0/t^{\beta\delta})$$

Since the correlation function for the Bose gas has a scaling form, we must ask what is wrong with the type of analysis described above and where does the second term come from. The answer lies in the fact that if the second order term in the expansion for the correlation length near the critical point ($\xi^{-1} \sim t^{\nu}[1 + t^{x} + \cdots]$) is included in the analysis, the correct form for $\mathcal{F}_s(\mu_0, T)$ will be obtained. Thus the term proportional to $t^{2-\alpha_s}$ in the free energy of the Bose gas when $d > 4$ is a correction to scaling that arises because of a second-order term in the expansion for the correlation length.

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